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Topological entanglement of polymers and Chern–Simons field theory

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Abstract. In recent times some interesting field-theoretical descriptions of the statistical mechanics of entangling polymers have been proposed by various authors. In these approaches, a single test polymer fluctuating in a background of static polymers or in a lattice of obstacles is considered. The extension to the case in which the configurations of two or more polymers become non-static is not straightforward unless their trajectories are severely constrained. In this paper we present another approach, based on Chern–Simons field theory, which is able to describe the topological entanglements of two fluctuating polymers in terms of gauge fields and second quantized replica fields.

1. Introduction

In recent times, some interesting field-theoretical descriptions of the statistical mechanics of polymer rings subjected to topological constraints have been proposed by various authors [1–9]. In all of these approaches, which are based on the pioneering works [10, 11], the inclusion of higher-order link invariants [12–14] is as yet an unsolved problem. On the other hand, these invariants are necessary in order to specify, in a unique way, the distinct topological states of the polymers. The main difficulty in the application of higher-order link invariants is that they cannot be expressed easily in terms of the variables which characterize the polymer, i.e. its trajectory in the three-dimensional (3D) space and its contour length. For this reason, in all analytical methods used to study entangling polymers the simplest topological invariant is considered, namely the Gauss linking number [5]. Even in this approximation, a rigorous treatment of the 3D entanglement problem is mathematically difficult. Basically, only the case of a single polymer fluctuating in a background of static polymers or fixed obstacles has been investigated until now. Alternatively, one chooses a test polymer and averages out in the partition sum the configurations of all the remaining chains. This is done by identifying the relevant collective variables for the problem and treating them as distributed in a Gaussian fashion [3, 4].

As has been suggested recently, for instance, in [5, 9], a possible way out of the above difficulties is the introduction of Chern–Simons (CS) field theories [15] in the treatment of polymer entanglement. Indeed, theories of this kind have already been applied successfully

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in related problems, such as, for instance, the self-entanglements of DNA molecules [16]. In this case, the geometric features of the system are captured by the so-called writhe, a non-topological number which can be reproduced in terms of CS amplitudes. However, when one considers the statistical mechanics of polymers, the writhe is replaced by the Gauss linking invariant and the situation becomes more complicated. As a matter of fact, in order to express this topological invariant by means of CS amplitudes, one has to remove the spurious contributions coming from the self-linking of the trajectories described by the polymers in the space. In pure CS field theories, defined on a simply connected manifold, it is possible to achieve this goal by introducing the so-called *framing* of the loops [13]. Unfortunately, this kind of regularization depends on the form of the trajectories. Thus, it introduces in the action of the polymers additional terms, which make the integration over their configurations very difficult.

In this paper, we treat the statistical mechanics of two fluctuating polymers subjected to topological entanglements. This is a physically interesting approximation of a system of polymers in which a particular molecule is identified and the rest of them are replaced by an ‘effective’ molecule [6]. To reproduce the topological term necessary to specify the topological states, the strategy is adopted to couple the polymers to two Abelian CS fields. All the undesired terms coming from the self-linking of the trajectories are eliminated without the introduction of *framing* by simply choosing the coupling constants in a suitable way. Of course, our procedure does not replace the *framing*, but it is sufficient to cancel the self-linking ambiguities at least for the CS amplitudes which are relevant in our context. As will be shown, the addition of CS fields allows the complete decoupling of the actions of the two polymers, so that it is possible to treat each of them separately with the powerful methods developed in [6, 7]. As an outcome, we are able to formulate the problem of two fluctuating polymers subjected to topological constraints in terms of a CS gauge field theory with n components at the limit in which n goes to zero. With respect to the other analytical approaches based on Edwards’ work [10], the external magnetic fields generated by the background polymers are replaced in our case by quantum CS fields. In this way, the somewhat artificial dependence on the conformations of the static polymers disappears from the partition function of the system. Moreover, let us stress that the CS fields enter in our formalism not just as auxiliary fields, but play an important physical role, since they propagate the forces which constrain the system in a given topological state.

The material presented in this paper is divided as follows. In section 2 we briefly review the field-theoretical approach developed in [6, 7] in the case of a test polymer entangling with another polymer of fixed conformation. In section 3 this approach will be extended to the situation in which both polymers are dynamical. In particular, the configurational probabilities derived in [7] are generalized to the case of two fluctuating polymers. Finally, in the conclusions, some possible generalizations and applications of our treatment will be discussed.

2. Statistical-mechanical theory of polymer entanglement

Let P be a polymer (see, e.g., [2, 9] for a general introduction to the physics of polymers) represented as a long chain of $N + 1$ segments $\vec{r}_{i+1} - \vec{r}_i$ for $i = 0, \dots, N$. Each segment has a fundamental step length a, which is assumed to be very small with respect to the total length of the polymer. Moreover, the junction between adjacent segments is such that they can rotate freely in all directions. In the limit of large values of N, the ensemble of M polymers $P_1, \dots, P_M$ of this kind can be regarded as the ensemble of M particles subjected to a self-avoiding random walk. The whole configuration of a polymer P is thus entirely specified by the trajectory $\boldsymbol{r}(s)$ of a particle in three-dimensional (3D) space, with $0 \leqslant s \leqslant L$. L is the contour length of the polymer and plays the role of time. We assume that the molecules of the polymer repel each other with a self-avoiding potential $v(\boldsymbol{r} - \boldsymbol{r}')$. The potential v must be strong enough to avoid unwanted intersections of the trajectory $\boldsymbol{r}(s)$ with itself. In the following, the case of two polymers P_1 and P_2 of contour length L_1 and L_2, respectively, will be investigated. We suppose that they describe the curves C_1 and C_2 in 3D space.

In order to take into account the entanglement of P_1 around P_2, we introduce the Gauss linking invariant $\chi(C_1, C_2)$,

$$\chi(C_1, C_2) \equiv \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} \mathrm{d}\boldsymbol{r}_1 \times \mathrm{d}\boldsymbol{r}_2 \cdot \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}. \tag{1}$$

For the moment, we confine ourselves to the situation in which one single test polymer, for instance P_1, is entangling with a polymer P_2 of static configuration $\boldsymbol{r}_2(s_2)$, with $0 \leqslant s_2 \leqslant L_2$. Let $G_m(\boldsymbol{r}_1, L_1; \boldsymbol{r}_{0,1}, 0)$ be the configurational probability to find P_1 with one end at a point $\boldsymbol{r}_1$ starting the other end at $\boldsymbol{r}_{0,1}$ and penetrating a number m of times a surface S_2 whose boundary is given by P_2. $G_m(\boldsymbol{r}_1, L_1; \boldsymbol{r}_{0,1}, 0)$ can be expressed in terms of path integrals as the Green function of a particle subjected to a self-avoiding random walk [7]

$$\begin{aligned} G_m(\boldsymbol{r}_1, L_1; \boldsymbol{r}_{0,1}, 0) = {}& \int_{\boldsymbol{r}_{0,1}=\boldsymbol{r}_1(0)}^{\boldsymbol{r}_1=\boldsymbol{r}_1(L_1)} \mathcal{D}\boldsymbol{r}_1(s_1)\, \delta(\chi(C_1, C_2) - m) \\ & \times \exp\left\{ -\int_0^{L_1} \mathrm{d}s_1\, \mathcal{L}_0 - \frac{1}{2a^2} \int_0^{L_1} \mathrm{d}s_1 \int_0^{L_1} \mathrm{d}s_1'\, v(\boldsymbol{r}(s_1) - \boldsymbol{r}(s_1')) \right\} \end{aligned} \tag{2}$$

where

$$\mathcal{L}_0 = \frac{3}{2a} \dot{\boldsymbol{r}}_1^2. \tag{3}$$

We remark here that the Gauss linking invariant takes integer values only if the polymers form closed rings. This case can be recovered easily, by requiring that $\boldsymbol{r}_1 = \boldsymbol{r}_{0,1}$ in the path integral (2). Due to the importance of open polymer chains in understanding the nature of topological forces, however, we will consider in the following also the more general situation in which $\boldsymbol{r}_1 \neq \boldsymbol{r}_{0,1}$.

To simplify the computations, it is convenient to work with the chemical potential λ conjugated to the topological charge m. Thus, we take the Fourier transform of $G_m(\boldsymbol{r}_1, L_1; \boldsymbol{r}_{0,1}, 0)$ with respect to m,

$$G_m(\boldsymbol{r}_1, L_1; \boldsymbol{r}_{0,1}, 0) = \int \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{-\mathrm{i}\lambda m} G_\lambda(\boldsymbol{r}_1, L_1; \boldsymbol{r}_{0,1}, 0). \tag{4}$$

Comparing with equation (2), the Green function $G_\lambda(\boldsymbol{r}_1, L_1; \boldsymbol{r}_{0,1}, 0)$ is given by

$$\begin{aligned} G_\lambda(\boldsymbol{r}_1, L_1; \boldsymbol{r}_{0,1}, 0) = {}& \int_{\boldsymbol{r}_{0,1}}^{\boldsymbol{r}_1} \mathcal{D}\boldsymbol{r}_1(s_1) \exp\left\{ -\int_0^{L_1} \mathrm{d}s_1 \mathcal{L}_0 \right\} \\ & \times \exp\left\{ -\frac{1}{2a^2} \int_0^{L_1} \mathrm{d}s_1 \int_0^{L_1} \mathrm{d}s_1'\, v(\boldsymbol{r}(s_1) - \boldsymbol{r}(s_1')) + \mathrm{i}\lambda\chi(C_1, C_2) \right\}. \end{aligned} \tag{5}$$

Following [10], the above configurational probability can be converted to that of a quantum particle diffusing in a magnetic field produced by the polymer P_2. To this end, we define at this point the current density

$$\boldsymbol{j}(\boldsymbol{r}_2) = \oint_{C_2} \dot{\boldsymbol{r}}_2(s_2)\, \delta(\boldsymbol{r} - \boldsymbol{r}_2(s_2)). \tag{6}$$

This current generates a magnetic field

$$B(r_1) = \frac{1}{4\pi}\oint_{C_2} \mathrm{d}r_2(s_2) \times \frac{r_1 - r_2(s_2)}{|r_1 - r_2(s_2)|^3} \tag{7}$$

which satisfies the relations $\nabla \cdot B = 0$ and $\nabla \times B = j_2$. In terms of B, the Gauss linking invariant (1) becomes

$$\chi(C_1, C_2) = \oint_{C_1} \mathrm{d}r_1(s_1) \cdot B(r_1(s_1)). \tag{8}$$

Let us note that the path integral (5) does not describe a Markoffian random walk due to the presence of the non-local self-avoiding term. To reduce it to a Markoffian random walk, we introduce Gaussian scalar fields $\phi(r)$ with propagator

$$\langle \phi(r)\,\phi(r')\rangle = \frac{1}{a^2} v(r - r'). \tag{9}$$

Thus, we have from equation (5)

$$G_\lambda(r_1, L_1; r_{0,1}, 0) = \langle G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B)\rangle_\phi. \tag{10}$$

In the above equation the symbol $\langle\ \rangle_\phi$ denotes the average over the auxiliary fields ϕ and

$$G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B) = \int_{r_{0,1}}^{r_1} \mathcal{D}r_1(s_1) \exp\left\{-\int_0^{L_1} \mathrm{d}s_1(\mathcal{L}_\phi + \mathrm{i}\dot{r}_1(s_1) \cdot \lambda B(r_1(s_1)))\right\} \tag{11}$$

where we have put

$$\mathcal{L}_\phi = \mathcal{L}_0 + \mathrm{i}\phi(r_1(s_1)). \tag{12}$$

The Green function $G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B)$ is formally that of a particle diffusing under the magnetic field B, defined in equation (7), and an imaginary electric potential $\mathrm{i}\phi$. Thus, it can be shown to satisfy the Schrödinger-like equation

$$\left\{\frac{\partial}{\partial L_1} - \frac{a}{6}(\nabla + \mathrm{i}\lambda B)^2 + \mathrm{i}\phi\right\} G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B) = \delta(L_1)\,\delta(r_1 - r_{0,1}). \tag{13}$$

The Laplace transform of the above equation with respect to the contour length L_1 is

$$\left\{z_1 - \tfrac{1}{6}a(\nabla + \mathrm{i}\lambda B)^2 + \mathrm{i}\phi\right\} G_\lambda(r_1, r_{0,1}; z_1|\phi, B) = \delta(r_1 - r_{0,1}) \tag{14}$$

where

$$G_\lambda(r_1, r_{0,1}; z_1|\phi, B) = \int_0^\infty \mathrm{d}L_1\, \mathrm{e}^{-z_1 L_1}\, G_\lambda(r_1, L_1; r_{0,1}, 0|\phi, B). \tag{15}$$

The variable z_1 plays the role of the chemical potential conjugate to the contour length L_1. From now on, we set $D = \nabla + \mathrm{i}\lambda B$. Starting from equation (14) and integrating over the auxiliary fields ϕ by means of the replica method, one can express $G_\lambda(r_1, r_{0,1}; z_1)$ in terms of second quantized fields. Skipping the details of the derivation, which can be found in [7], we just state the result obtained in the case of a self-avoiding potential of the kind $v(r) = a^2 v_0\,\delta(r)$,

$$G_\lambda(r_1, r_{0,1}; z_1) = \lim_{n\to 0}\int \prod_{\omega=1}^{n} \mathcal{D}\psi^{*\omega}\,\mathcal{D}\psi^{\omega}\ \psi^{*\overline{\omega}}(r_1)\,\psi^{\overline{\omega}}(r_{0,1})\,\mathrm{e}^{-F[\Psi]}. \tag{16}$$

In the above equation the fields $\psi^{*\omega}, \psi^{\omega}$, $\omega = 1, \ldots, n$, are complex replica fields and $\Psi = (\psi^1, \ldots, \psi^n)$. Moreover, $\overline{\omega}$ is an arbitrarily chosen integer in the range $1, \ldots, n$ and the polymer free energy $F[\Psi]$ is given by

$$F[\Psi] = \int \mathrm{d}^3 r \left\{\tfrac{1}{6}a\|D\Psi\|^2 + z_1\|\Psi\|^2 + v_0\|\Psi\|^4\right\} \tag{17}$$

where $\|\boldsymbol{D}\Psi\|^2 = \sum_\omega (\boldsymbol{D}\psi^\omega)^\dagger \boldsymbol{D}\psi^\omega$ and $\|\Psi\|^2 = \sum_\omega |\psi^\omega|^2$. The generalization of the above formulae to an arbitrary number M of static polymers $P_2, \dots, P_M$ is straightforward, but not the inclusion of their fluctuations. Already, in the case of two polymers, the analysis of the Schrödinger equation (14) becomes complicated due to the presence of the non-trivial interactions among the polymers introduced by the Gauss linking invariant (1). This makes the derivation of the Green function $G_\lambda(\boldsymbol{r}_1, L_1; \boldsymbol{r}_{0,1}, 0|\phi, \boldsymbol{B})$ in terms of second quantized fields extremely difficult, apart from a few cases in which the trajectories of the polymers are strongly constrained. On the other hand, without including the fluctuations of all polymers, there is always the difficulty of determining those configurations of the static polymers which are physically relevant. Indeed, we see from equations (16) and (17) that the free energy $F[\Psi]$ of the test polymer P_1 depends on the trajectory $\boldsymbol{r}_2(s)$ through the external magnetic potential $\boldsymbol{B}$ contained in the covariant derivative $\boldsymbol{D}$. As we will see in the next section, the introduction of auxiliary Chern–Simons fields will remove all these problems.

3. Topological entanglement of polymers via Chern–Simons fields

We study in this section the fluctuations of two polymers P_1 and P_2 subjected to topological constraints. In analogy with the previous section, we consider the configurational probability $G_m(\{\boldsymbol{r}\}, \{L\}; \{\boldsymbol{r}_0\}, 0)$ of finding the polymer P_1 with ends in $\boldsymbol{r}_1$ and $\boldsymbol{r}_{0,1}$ and the polymer P_2 with ends in $\boldsymbol{r}_2$ and $\boldsymbol{r}_{0,2}$. Moreover, we require that P_1 winds up around P_2 a number m times. Here, we have put $\{\boldsymbol{r}\} = \boldsymbol{r}_1, \boldsymbol{r}_2$, $\{L\} = L_1, L_2$, etc. From now on, the indices $\tau, \tau', \dots = 1, 2$ will be used to distinguish between the two different polymers. The self-avoiding potential of the previous section must be extended in the present case in order to take into account the reciprocal repulsions among the molecules of the two different polymers. Thus we choose a two-body potential of the kind

$$v_{\tau\tau'}(\boldsymbol{r}_\tau(s_\tau) - \boldsymbol{r}_{\tau'}(s'_{\tau'})) = v^0_{\tau\tau'} v(\boldsymbol{r}_\tau(s_\tau) - \boldsymbol{r}_{\tau'}(s'_{\tau'})) \tag{18}$$

where $v^0_{\tau\tau'}$ is a symmetric 2×2 matrix and $v(\boldsymbol{r})$ is strongly repulsive. As seen in the previous section, the presence of self-avoiding potentials leads to random walks which are not Markoffian. To solve this problem, we introduce auxiliary scalar fields with Gaussian action and propagator

$$\langle \phi_\tau(\boldsymbol{x}) \phi_{\tau'}(\boldsymbol{y}) \rangle = \frac{1}{a^2} v^0_{\tau\tau'} v(\boldsymbol{x} - \boldsymbol{y}). \tag{19}$$

In the future we will make use of the following formula:

$$\int \prod_{\tau=1}^{2} \mathcal{D}\phi_\tau \exp\left\{ -\tfrac{1}{2} a^2 \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{y}\, \phi_\tau(\boldsymbol{x}) M^{\tau\tau'}(\boldsymbol{x}, \boldsymbol{y})\, \phi_{\tau'}(\boldsymbol{y}) - \mathrm{i} \sum_{\tau=1}^{2} \int \mathrm{d}^3\boldsymbol{x}\, J_\tau(\boldsymbol{x})\, \phi_\tau(\boldsymbol{x}) \right\}$$
$$= \exp\left\{ -\frac{1}{2a^2} \int_0^{L_\tau} \int_0^{L_{\tau'}} \mathrm{d}s_\tau\, \mathrm{d}s'_{\tau'} v_{\tau\tau'}(\boldsymbol{r}_\tau(s_\tau) - \boldsymbol{r}_{\tau'}(s'_{\tau'})) \right\} \tag{20}$$

where $M^{\tau\tau'}(\boldsymbol{x}, \boldsymbol{y})$ is the inverse of the matrix $v_{\tau\tau'}(\boldsymbol{x} - \boldsymbol{y})$ and

$$J_\tau(\boldsymbol{x}) = \int_0^{L_\tau} \mathrm{d}s_\tau\, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{r}_\tau(s_\tau)). \tag{21}$$

Let us now rewrite the topological contribution in the path integral (5) in a more suitable way by means of auxiliary CS fields. With the introduction of these fields, our treatment of the polymer entanglement problem departs from that of section 2 and from [7].

We will consider for our purposes Abelian CS field theories of action

$$\mathcal{A}_{CS}(\boldsymbol{A}, \kappa) = \frac{\kappa}{8\pi} \int \mathrm{d}^3\boldsymbol{x}\, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{22}$$

with $\mu, \nu, \rho = 1, 2, 3$. κ is a real coupling constant and $\epsilon^{\mu\nu\rho}$ is the completely antisymmetric tensor in 3D. The above action can also be written in another useful form:

$$\mathcal{A}_{CS}(\boldsymbol{A}, \kappa) = \frac{\kappa}{8\pi} \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) \tag{23}$$

where $\boldsymbol{r} = (x^1, x^2, x^3)$. To quantize the CS theory we choose the Feynman gauge with propagator

$$G_{\mu\nu}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\mathrm{i}}{\kappa}\epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|\boldsymbol{x}-\boldsymbol{y}|^3}. \tag{24}$$

The observables of the theory are gauge-invariant operators built out of the basic fields A_μ. A complete set is given by the holonomies around closed curves

$$\mathcal{W}(C, \gamma) \equiv \exp\left\{-\mathrm{i}\gamma \oint_C A_\mu\, \mathrm{d}x^\mu\right\}. \tag{25}$$

The vacuum expectation value of two of these observables $\mathcal{W}(C_1, \gamma_1)$ and $\mathcal{W}(C_2, \gamma_2)$ is

$$\begin{aligned}&\langle \mathcal{W}(C_1, \gamma_1)\, \mathcal{W}(C_2, \gamma_2)\rangle_A \\ &\quad = \exp\left\{-\mathrm{i}\left(\frac{2\pi}{\kappa}\right)\left[\gamma_1^2\chi(C_1, C_1) + \gamma_2^2\chi(C_2, C_2) + 2\gamma_1\gamma_2\chi(C_1, C_2)\right]\right\}\end{aligned} \tag{26}$$

where $\chi(C_\tau, C_\tau)$, $\tau = 1, 2$ is the so-called self-linking number of the loop C_τ.

To reproduce the term of equation (5) containing the Gauss invariant χ, we need two Chern–Simons fields a_μ and b_μ with actions $\mathcal{A}_{CS}(\boldsymbol{a}, \kappa)$ and $\mathcal{A}_{CS}(\boldsymbol{b}, -\kappa)$, respectively. Using equation (26) and setting, for instance,

$$\gamma_1 = \frac{\kappa}{4\pi} \qquad \gamma_2 = \frac{\lambda}{4} \tag{27}$$

one sees, in fact, that

$$\langle \mathcal{W}(C_1, \gamma_1)\, \mathcal{W}(C_2, \gamma_2)\rangle_a \qquad \langle \mathcal{W}(C_1, -\gamma_1)\, \mathcal{W}(C_2, \gamma_2)\rangle_b = \exp\{-\mathrm{i}\lambda\chi(C_1, C_2)\}. \tag{28}$$

The right-hand side of the above equation is exactly the contribution due to the topological entanglements of the polymers appearing in equation (5). We are now ready to write the expression of the Green function $G_\lambda(\{\boldsymbol{r}\}, \{L\}; \{\boldsymbol{r}_0\}, 0)$ for two entangling polymers. First of all, let us put

$$G_\lambda(\{\boldsymbol{r}\}, \{L\}; \{\boldsymbol{r}_0\}, 0) = \langle G_\lambda(\{\boldsymbol{r}\}, \{L\}; \{\boldsymbol{r}_0\}, 0|\{\phi\}, \{\boldsymbol{A}\})\rangle_{\{\phi\},\boldsymbol{a},\boldsymbol{b}} \tag{29}$$

where $\langle\ \rangle_{\{\phi\},\boldsymbol{a},\boldsymbol{b}}$ denotes the average with respect to the fields $\phi_\tau, \boldsymbol{a}, \boldsymbol{b}$ and

$$\begin{aligned}G_\lambda(\{\boldsymbol{r}\}, \{L\}; \{\boldsymbol{r}_0\}, 0|\{\phi\}, \{\boldsymbol{A}\}) &= \prod_{\tau=1}^{2}\int_{\boldsymbol{r}_{0,\tau}}^{\boldsymbol{r}_\tau} \mathcal{D}\boldsymbol{r}_\tau(s_\tau) \exp\left\{-\int_0^{L_\tau} \mathcal{L}_{\phi_\tau}\right\} \\ &\quad\times \exp\left\{-\mathrm{i}\gamma_\tau \int_0^{L_\tau} \boldsymbol{A}^\tau(\boldsymbol{r}_\tau(s_\tau)) \cdot \mathrm{d}\boldsymbol{r}_\tau(s_\tau)\right\}.\end{aligned} \tag{30}$$

The parameters γ_τ appearing in the above equation are defined as in equation (27) and the fields $\boldsymbol{A}^\tau$ are given by the relation

$$\boldsymbol{A}^\tau = \boldsymbol{a} + (-1)^\tau \boldsymbol{b} \qquad \tau = 1, 2. \tag{31}$$

To show that equation (29) provides the desired generalization of equation (2) to the case of two fluctuating polymers, we exploit formulae (20) and (28) in order to perform the two independent integrations over the fields ϕ_τ, $\boldsymbol{a}$ and $\boldsymbol{b}$. After some calculations one finds that

$$G_\lambda(\{\boldsymbol{r}\},\{L\};\{\boldsymbol{r}_0\},0)=\int_{\boldsymbol{r}_{0,1}}^{\boldsymbol{r}_1}\mathcal{D}\boldsymbol{r}_1(s_1)\int_{\boldsymbol{r}_{0,2}}^{\boldsymbol{r}_2}\mathcal{D}\boldsymbol{r}_2(s_2) \tag{32}$$

$$\begin{aligned}&\times\exp\left\{-\int_0^{L_1}\mathrm{d}s_1\,\mathcal{L}_0(\dot{\boldsymbol{r}}_1(s_1))-\int_0^{L_2}\mathrm{d}s_2\,\mathcal{L}_0(\dot{\boldsymbol{r}}_2(s_2))\right\}\\&\times\exp\left\{-\frac{1}{2a^2}\sum_{\tau,\tau'=1}^{2}\int_0^{L_\tau}\mathrm{d}s_\tau\int_0^{L_{\tau'}}\mathrm{d}s_{\tau'}\,v_{\tau\tau'}(\boldsymbol{r}_\tau(s_\tau)-\boldsymbol{r}_{\tau'}(s_{\tau'}))-\mathrm{i}\lambda\chi(C_1,C_2)\right\}.\end{aligned} \tag{33}$$

By inverse Fourier transformation in λ as in equation (4), we obtain from equation (33):

$$\begin{aligned}G_m(\{\boldsymbol{r}\},\{L\};\{\boldsymbol{r}_0\},0)=&\int_{\boldsymbol{r}_{0,1}}^{\boldsymbol{r}_1}\mathcal{D}\boldsymbol{r}_1(s_1)\int_{\boldsymbol{r}_{0,2}}^{\boldsymbol{r}_2}\mathcal{D}\boldsymbol{r}_2(s_2)\\&\times\exp\left\{-\int_0^{L_1}\mathrm{d}s_1\,\mathcal{L}_0(\dot{\boldsymbol{r}}_1(s_1))-\int_0^{L_2}\mathrm{d}s_2\,\mathcal{L}_0(\dot{\boldsymbol{r}}_2(s_2))\right\}\\&\times\exp\left\{-\frac{1}{2a^2}\sum_{\tau,\tau'=1}^{2}\int_0^{L_\tau}\mathrm{d}s_\tau\int_0^{L_{\tau'}}\mathrm{d}s'_{\tau'}v_{\tau\tau'}(\boldsymbol{r}_\tau(s_\tau)-\boldsymbol{r}_{\tau'}(s'_{\tau'}))\right\}\\&\times\delta(\chi(C_1,C_2)-m).\end{aligned} \tag{34}$$

If we ignore the fluctuations of P_2 and the reciprocal repulsion among the molecules of P_1 and P_2, which was not taken into account in section 2, equation (34) coincides exactly with equation (2) as desired. With respect to equation (34), the formulation of the polymer problem given in equation (29) in terms of Chern–Simons fields presents the advantage that now each polymer P_1 and P_2 undergoes an independent random walk. Their mutual interactions, that in equations (2) and (34) occur through the Gauss linking invariant $\chi(C_1,C_2)$, are now mediated by Chern–Simons fields, as it is possible to see from equations (29) and (30). At this point, it becomes possible to express the Green function $G_\lambda(\{\boldsymbol{r}\},\{L\},\{\boldsymbol{r}_0\},0)$ in terms of second quantized fields using the methods developed in [7]. To this end, we split the Green function $G_\lambda(\{\boldsymbol{r}\},\{L\},\{\boldsymbol{r}_0\},0|\{\phi\},\{\boldsymbol{A}\})$ of equation (30) as follows:

$$G_\lambda(\{\boldsymbol{r}\},\{L\};\{\boldsymbol{r}_0\},0|\{\phi\},\{\boldsymbol{A}\})=G_\lambda(\boldsymbol{r}_1,L_1;\boldsymbol{r}_{0,1},0|\phi_1,\boldsymbol{A}^1)\,G_\lambda(\boldsymbol{r}_2,L_2;\boldsymbol{r}_{0,2},0|\phi_2,\boldsymbol{A}^2) \tag{35}$$

where, for $\tau=1,2$:

$$G_\lambda(\boldsymbol{r}_\tau,L_\tau;\boldsymbol{r}_{0_\tau},0|\phi_\tau,\boldsymbol{A}^\tau)=\int_{\boldsymbol{r}_{0,\tau}}^{\boldsymbol{r}_\tau}\mathcal{D}\boldsymbol{r}_\tau(s_\tau)\exp\left\{-\int_0^{L_\tau}\left[\mathcal{L}_{\phi_\tau}-\mathrm{i}\gamma_\tau\boldsymbol{A}^\tau\cdot\mathrm{d}\boldsymbol{r}_\tau(s_\tau)\right]\right\}. \tag{36}$$

Each Green function $G_\lambda(\boldsymbol{r}_\tau,L_\tau;\boldsymbol{r}_{0,\tau},0|\phi_\tau,\boldsymbol{A}^\tau)$, $\tau=1,2$, is the Green function of a particle diffusing under the vector potential $\boldsymbol{A}^\tau$ and the imaginary electromagnetic field ϕ_τ. As a consequence, it can be written as the solution of a Schrödinger-like equation such as (13). In analogy with the previous section, it is convenient to introduce chemical potentials z_τ conjugated to L_τ. Thus, we consider the Laplace transformed Green function

$$G_\lambda(\{\boldsymbol{r}\},\{\boldsymbol{r}_0\};\{z\}|\{\phi\},\{\boldsymbol{A}\})=\int_0^\infty\int_0^\infty\mathrm{d}L_1\,\mathrm{d}L_2\,\mathrm{e}^{-(z_1L_1+z_2L_2)}G_\lambda(\{\boldsymbol{r}\},\{L\};\{\boldsymbol{r}_0\},0|\phi_\tau,\boldsymbol{A}^\tau). \tag{37}$$

From equations (35) and (36) we have

$$G_\lambda(\{\boldsymbol{r}\},\{\boldsymbol{r}_0\};\{z\}|\{\phi\},\{\boldsymbol{A}\})=G_\lambda(\boldsymbol{r}_1,\boldsymbol{r}_{0,1};z_1|\phi_1,\boldsymbol{A}^1)\,G_\lambda(\boldsymbol{r}_2,\boldsymbol{r}_{0,2};z_2|\phi_2,\boldsymbol{A}^2) \tag{38}$$

where the functions $G_\lambda(\boldsymbol{r}_\tau, \boldsymbol{r}_{0,\tau}; z_\tau|\phi_\tau, \boldsymbol{A}^\tau)$ have already been defined in equation (15). For each value of $\tau = 1, 2$, they explicitly satisfy equation (14), which is shown again here, for convenience:

$$\left\{z_\tau - \tfrac{1}{6}a\boldsymbol{D}_\tau^2 + \mathrm{i}\phi_\tau\right\}G_\lambda(\boldsymbol{r}_\tau, \boldsymbol{r}_{0,\tau}; z_\tau|\phi_\tau, \boldsymbol{A}^\tau) = \delta(\boldsymbol{r}_\tau - \boldsymbol{r}_{0,\tau}). \tag{39}$$

The covariant derivatives $\boldsymbol{D}_\tau$ are defined as follows: $\boldsymbol{D}_\tau = \boldsymbol{\nabla} + \mathrm{i}\gamma_\tau\boldsymbol{A}^\tau$. The solution of equation (39) in terms of complex scalar fields ψ_τ^*, ψ_τ is

$$G_\lambda(\boldsymbol{r}_\tau, \boldsymbol{r}_{0,\tau}; z_\tau|\phi_\tau, \boldsymbol{A}^\tau) = \frac{1}{Z_\tau}\int \mathcal{D}\psi_\tau^*\,\mathcal{D}\psi_\tau\;\psi_\tau^*(\boldsymbol{r}_\tau)\,\psi_\tau(\boldsymbol{r}_{0,\tau})\,\mathrm{e}^{-F[\psi_\tau]} \tag{40}$$

where, setting $|\boldsymbol{D}_\tau\psi_\tau|^2 = (\boldsymbol{D}_\tau\psi_\tau)^\dagger\cdot\boldsymbol{D}_\tau\psi_\tau$, the free energy $F[\psi_\tau]$ is given by

$$F[\psi_\tau] = \int \mathrm{d}^3\boldsymbol{r}\left[\tfrac{1}{6}a|\boldsymbol{D}_\tau\psi_\tau|^2 + (z_\tau + \mathrm{i}\phi_\tau)|\psi_\tau|^2\right]. \tag{41}$$

Finally, the partition function Z_τ is

$$Z_\tau = \int \mathcal{D}\psi_\tau^*\,\mathcal{D}\psi_\tau\,\mathrm{e}^{-F[\psi_\tau]}. \tag{42}$$

As we see from equation (41), $F[\psi]$ is nothing but the Gintzburg–Landau free energy of a superconductor in a fluctuating magnetic field. We are now ready to perform the average over the auxiliary fields ϕ_τ in the Green function (38). This integration is, however, highly non-trivial. As a matter of fact, using equation (40) to express the original Green function (38) in terms of second quantized fields, one immediately realizes that the integrand is not Gaussian, due to the presence of the factors Z_τ^{-1}. To solve this problem, we exploit the replica method. Thus we introduce $2n$ replica fields $\psi_\tau^{*\omega}, \psi_\tau^\omega$, with $\tau = 1, 2$ and $\omega = 1, \ldots, n$. In terms of these fields, the Green function (38) can be written as follows:

$$G_\lambda(\{\boldsymbol{r}\}, \{\boldsymbol{r}_0\}; \{z\}|\{\phi\}, \{\boldsymbol{A}\}) = \lim_{n\to 0}\prod_{\tau=1}^{2}\left[\int\prod_{\omega=1}^{n}\mathcal{D}\psi_\tau^{*\omega}\,\mathcal{D}\psi_\tau^{\omega}\;\psi_\tau^{*\overline{\omega}}(\boldsymbol{r}_\tau)\,\psi_\tau^{\overline{\omega}}(\boldsymbol{r}_{0,\tau})\,\mathrm{e}^{-F[\psi_\tau^\omega]}\right] \tag{43}$$

where $\overline{\omega}$ is an arbitrary integer chosen in the range $1 \leqslant \overline{\omega} \leqslant n$. According to the replica method, we will also assume that the limit for n going to zero commutes with the integrations in the fields $\boldsymbol{A}^\tau$ and ϕ_τ. In this way, the integral over the auxiliary fields ϕ_τ becomes Gaussian and can be performed easily. Supposing in analogy with the previous section that the self-avoiding potential is of the kind

$$v_{\tau\tau'}(\boldsymbol{r}) = v_{\tau\tau'}^0\delta(\boldsymbol{r}) \tag{44}$$

we have after some calculations that

$$\begin{aligned}&\langle G_\lambda(\{\boldsymbol{r}\}, \{\boldsymbol{r}_0\}; \{z\}|\{\phi\}, \{\boldsymbol{A}\})\rangle_{\{\phi\},\boldsymbol{a},\boldsymbol{b}}\\ &\quad= \lim_{n\to 0}\int\mathcal{D}\boldsymbol{a}\,\mathcal{D}\boldsymbol{b}\prod_{\tau=1}^{2}\left\{\left[\prod_{\omega=1}^{n}\mathcal{D}\psi_\tau^{*\omega}\mathcal{D}\psi_\tau^{\omega}\right]\psi_\tau^{*\overline{\omega}}(\boldsymbol{r}_\tau)\psi_\tau^{\overline{\omega}}(\boldsymbol{r}_{0,\tau})\right\}\exp\{-\mathcal{A}(\boldsymbol{a}, \boldsymbol{b}, \{\Psi\})\}\end{aligned} \tag{45}$$

where, using the same notation of equations (16) and (17), the action $\mathcal{A}(\boldsymbol{a}, \boldsymbol{b}, \{\Psi\})$ is

$$\begin{aligned}\mathcal{A}(\boldsymbol{a}, \boldsymbol{b}, \{\Psi\}) &= \mathrm{i}\mathcal{A}_{CS}(\boldsymbol{a}, \kappa) + \mathrm{i}\mathcal{A}_{CS}(\boldsymbol{b}, -\kappa)\\ &\quad+\sum_{\tau=1}^{2}\left[\tfrac{1}{6}a\|\boldsymbol{D}_\tau\Psi_\tau\|^2 + z_\tau\|\Psi_\tau\|^2\right] + \sum_{\tau,\tau'=1}^{2}\|\Psi_\tau\|^2 v_{\tau\tau'}^0\|\Psi_{\tau'}\|^2.\end{aligned} \tag{46}$$

Equation (45) is the generalization of equation (16), which describes in terms of fields the configurational probability for two entangling polymers P_τ, $\tau = 1, 2$, to have their ends in

$\boldsymbol{r}_\tau$ and $\boldsymbol{r}_{0,\tau}$, respectively. Apparently, with respect to equation (29), the Green function (45) contains an extra parameter provided by the CS coupling constant κ. In fact, κ is not present in the first quantized formalism due to its cancellation occurring in (28). However, one can easily check the absence of κ in (29) by performing a simple linear transformation in the CS fields. To this end, we replace the fields $\boldsymbol{a}$ and $\boldsymbol{b}$ with the fields $\boldsymbol{A}^1$ and $\boldsymbol{A}^2$ putting in equation (45)

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{A}^1 \qquad \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{A}^2. \tag{47}$$

After some calculations, the action (46) becomes, in terms of $\boldsymbol{A}^\tau$,

$$\begin{aligned}\mathcal{A}(\boldsymbol{A}^1, \boldsymbol{A}^2, \{\Psi\}) = {} & \frac{\mathrm{i}}{16\pi}\int \mathrm{d}^3\boldsymbol{x}\, \epsilon^{\mu\nu\rho}(kA^1_\mu)\partial_\nu A^2_\rho \\ & + \tfrac{1}{6}a\left\|\left(\nabla + \mathrm{i}\frac{(k\boldsymbol{A}^1)}{4\pi}\right)\Psi_1\right\|^2 + \tfrac{1}{6}a\left\|\left(\nabla + \mathrm{i}\gamma_2\boldsymbol{A}^2\right)\Psi_2\right\|^2 \\ & + \sum_{\tau=1}^{2} z_\tau \|\Psi_\tau\|^2 + \sum_{\tau,\tau'=1}^{2} \|\Psi_\tau\|^2\, v^0_{\tau\tau'}\, \|\Psi_{\tau'}\|^2.\end{aligned} \tag{48}$$

As it is possible to see, in the new action (48) the parameter κ appears as a factor multiplying the field $\boldsymbol{A}^1$ and can thus be eliminated by a rescaling of these fields.

The configuration probability (45) is given in the space of the chemical potentials λ and z_τ conjugated to the topological number m and the contour lengths L_τ. It is also possible to find an expression of the above configurational probability in the space of the topological number m by taking the inverse Fourier transformation of the Green function (45)

$$G_m(\{\boldsymbol{r}\}, \{\boldsymbol{r}_0\}, \{z\}) = \int \frac{\mathrm{d}\lambda}{2\pi}\mathrm{e}^{-\mathrm{i}\lambda m}\langle G_\lambda(\{\boldsymbol{r}\}, \{\boldsymbol{r}_0\}, \{z\}|\{\phi\}, \{\boldsymbol{A}\})\rangle_{\{\phi\},\boldsymbol{a},\boldsymbol{b}}. \tag{49}$$

To this end, we split the action (46) into three parts

$$\mathcal{A}(\boldsymbol{a}, \boldsymbol{b}, \{\Psi\}) = \mathcal{A}_0(\{\boldsymbol{A}\}, \{\Psi\}) + \lambda\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{i}_2(\boldsymbol{r})\cdot\boldsymbol{A}^2(\boldsymbol{r}) + \tfrac{1}{6}a\lambda^2\int \tfrac{1}{16}\mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A}^2\cdot\boldsymbol{A}^2\|\Psi_2(\boldsymbol{r})\|^2 \tag{50}$$

where $\mathcal{A}_0(\{\boldsymbol{A}\}, \{\psi^\omega\})$ is the contribution to the action $\mathcal{A}(\boldsymbol{a}, \boldsymbol{b}, \{\psi^\omega\})$ which does not contain λ:

$$\begin{aligned}\mathcal{A}_0(\{\boldsymbol{A}\}, \{\Psi\}) = {} & \mathrm{i}\mathcal{A}_{CS}(\boldsymbol{a}, \kappa) + \mathrm{i}\mathcal{A}_{CS}(\boldsymbol{b}, -\kappa) + \tfrac{1}{6}a\|\boldsymbol{D}_1\Psi_1\|^2 + \tfrac{1}{6}a\|\boldsymbol{\nabla}\Psi_2\|^2 \\ & + \sum_{\tau=1}^{2} z_\tau\|\Psi_\tau\|^2 + \sum_{\tau,\tau'=1}^{2}\|\Psi_\tau\|^2 v^0_{\tau\tau'}\|\Psi_{\tau'}\|^2\end{aligned} \tag{51}$$

and

$$\boldsymbol{i}_2(\boldsymbol{r}) = \frac{a}{12}\frac{1}{2\mathrm{i}}\left(\Psi_2^*\nabla\Psi_2 - \Psi_2\nabla\Psi_2^*\right) = \frac{a}{12}\frac{1}{2\mathrm{i}}\sum_{\omega=1}^{n}(\psi_2^{*\omega}\nabla\psi_2^\omega - \psi_2\nabla\psi_2^{*\omega}). \tag{52}$$

Performing the Gauss integral in (49) and neglecting irrelevant constant factors, we have

$$\begin{aligned}G_m(\{\boldsymbol{r}\}, \{\boldsymbol{r}_0\}; \{z\}) = {} & \lim_{n\to 0}\int \mathcal{D}\boldsymbol{a}\,\mathcal{D}\boldsymbol{b}\prod_{\tau=1}^{2}\left[\prod_{\omega=1}^{n}\mathcal{D}\psi_\tau^{*\omega}\,\mathcal{D}\psi_\tau^\omega\;\psi_\tau^{*\overline{\omega}}(\boldsymbol{r}_\tau)\psi_\tau^{\overline{\omega}}(\boldsymbol{r}_{0,\tau})\right] \\ & \times\exp\left\{-\mathcal{A}_0(\{\boldsymbol{A}\}, \{\psi^\omega\})\right\} \\ & \times\exp\left\{-\tfrac{1}{4}K^{-1}\left(m - \mathrm{i}\sum_{\tau=1}^{2}\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{i}_2(\boldsymbol{r})\cdot\boldsymbol{A}^2(\boldsymbol{r})\right)^2\right\}K^{-1/2}\end{aligned} \tag{53}$$

where

$$K = \tfrac{1}{6}a\int \tfrac{1}{16}\mathrm{d}^3\boldsymbol{r}\,\boldsymbol{A}^2\cdot\boldsymbol{A}^2\|\Psi_2\|^2. \tag{54}$$

4. Conclusions

In this paper a CS-based model of polymers subjected to topological constraints has been derived and applied to the description of two entangling polymers P_1 and P_2. The final result is summarized by the Green function of equation (53). In the space of the chemical potential λ conjugated to the topological number m, the model consists in a generalization of a topological Ginzburg–Landau model in its n-component version (equations (45) and (46)). Similar theories in which the Lagrangian of Ginzburg–Landau is coupled to a single Abelian gauge field theory, have been considered in connection with superconductivity and liquid crystals [17].

As we have seen in section 3, the introduction of auxiliary CS fields has completely decoupled the actions of the two entangling polymers P_1 and P_2. In the original formulation of equation (34), in fact, the two polymers were coupled in a non-trivial way due to the reciprocal topological interactions introduced by the Gauss linking invariant. In equation (29), instead, each polymer P_τ, with $\tau = 1, 2$, interacts only with the auxiliary fields $\boldsymbol{A}_\tau, \phi_\tau$ and its action is formally that of a particle moving in the background of the 'electromagnetic' field $(\boldsymbol{A}^\tau, \mathrm{i}\phi_\tau)$. In our approach the Chern–Simons fields acquire the nice physical role of carriers of the topological interactions that constrain the polymer system to remain in a given topological state. In particular, it would be tempting to identify the CS fields as propagators of the collective modes which are relevant in the topological entanglements of polymers. A set of these collective models has been derived in [3].

On the other hand, the complications introduced in our formalism by the fact that both polymers are non-static, are minimal. To show this, we have generalized all the polymer configurational probabilities derived in [7] to the case of two fluctuating polymers. One can for instance see that, apart from the coupling with the quantum CS fields $\boldsymbol{a}$ and $\boldsymbol{b}$, which replace the external magnetic field $\boldsymbol{B}$ and eliminate the somewhat artificial dependence on the configurations of the static polymers, the Green function in equation (45) differs from that of equation (16) only by the presence of two sets of replica fields instead of one.

Of course, the inclusion of CS fields in the treatment of polymers also opens the possibility of taking into account more sophisticated topological invariants than the Gauss linking number, but this extension is not straightforward and requires some care. For instance, after replacing the fields $\boldsymbol{a}$ and $\boldsymbol{b}$ with their non-Abelian counterparts, one can obtain higher-order knot invariants from the radiative corrections of equation (28) as shown in [18]. However, in the non-Abelian case the elimination of the undesired Gauss self-linking terms occurring in equation (28) is valid only at the first-order approximation with respect to the CS coupling constant κ and remains a problem at higher orders. A possible solution to this difficulty is the introduction of a suitable *framing* such that

$$\chi_{\mathrm{framed}}(C, C) = 0. \tag{55}$$

Unfortunately, as briefly mentioned in the introduction, a *framing*, like that in equation (55), is necessarily, depending on the form of the loop C which, in turn, is a dynamical variable in the present context. Thus, the choice of *framing* would terribly complicate the form of the Schrödinger equation (39), preventing its solution in terms of second quantized fields.

In conclusion, we hope that it will be possible to extend the present approach to the situation in which there is an arbitrary number of polymers. We also note that in the configurational probability (53) the effects of self-entanglements of the loops have not been taken into account. However, these effects turn out to have relevant consequences in the statistics of the polymers and can be expressed in terms of Chern–Simons amplitudes as shown in [4]. The incorporation of self-entanglement within our approach is currently a problem under study. Finally, we remark that our results could also be applied to other physical systems such as vortex rings and dislocation lines embedded in a solid, in which topological constraints among entangled one-dimensional excitations in a continuum play essential roles.

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References

[1] Cardy J 1994 *Phys. Rev. Lett.* **72** 1580
Nechaev S K and Rostiashvili V G 1993 *J. Physique II* **3** 91
[2] Kholodenko A L and Vilgis T A 1998 *Phys. Rep.* **298** 251
[3] Brereton M G and Vilgis T A 1995 *J. Phys. A: Math. Gen.* **28** 1149
[4] Brereton M G 1995 *J. Mol. Struct. (Theochem.)* **336** 191
[5] Otto M and Vilgis T A 1996 *J. Phys. A: Math. Gen.* **29** 3893
Nechaev S 1990 *Int. J. Mod. Phys.* B **4** 1809
[6] Brereton M G and Shah S 1980 *J. Phys. A: Math. Gen.* **13** 2751
Brereton M G and Shah S 1981 *J. Phys. A: Math. Gen.* **14** L51
Brereton M G and Shah S 1982 *J. Phys. A: Math. Gen.* **15** 989
Elderfield D J 1982 *J. Phys. A: Math. Gen.* **15** 1369
[7] Tanaka F 1982 *Prog. Theor. Phys.* **68** 148
[8] Tanaka F 1982 *Prog. Theor. Phys.* **68** 164
[9] Kleinert H 1995 *Path Integrals* 2nd edn (Singapore: World Scientific)
[10] Edwards S F 1967 *Proc. Phys. Soc.* **91** 513
Edwards S F 1968 *J. Phys. A: Math. Gen.* **1** 15
[11] de Gennes P G 1972 *Phys. Lett.* A **38** 339
des Cloiseaux J 1974 *Phys. Rev.* A **10** 1665
Emery V J 1975 *Phys. Rev.* B **11** 239
[12] Kauffmann L H 1993 *Knots and Physics* (Singapore: World Scientific)
[13] Witten E 1989 *Commun. Math. Phys.* **121** 351
[14] Jones V F R 1989 *Bull. Am. Math. Soc.* **129** 103
[15] Jackiw R and Templeton S 1981 *Phys. Rev.* D **23** 2291
Deser S, Jackiw R and Templeton S 1983 *Phys. Rev. Lett.* **48** 975
Schonfeld J 1981 *Nucl. Phys.* B **185** 157
Hagen C R 1984 *Ann. Phys.* **157** 342
[16] Moroz J D and Kamien R D 1997 *Nucl. Phys.* B **506** 695
[17] de Gennes P G 1974 *The Physics of Liquid Crystals* (Oxford: Oxford University Press)
[18] Guadagnini E, Martellini M and Mintchev M 1990 *Nucl. Phys.* B **336** 581
Labastida J M F and Ramallo A V 1989 *Phys. Lett.* B **238** 214